

Finite descent obstruction and non-abelian reciprocity

Otto Overkamp *

Abstract

For a nice algebraic variety X over a number field F , one of the central problems of Diophantine Geometry is to locate precisely the set $X(F)$ inside $X(\mathbf{A}_F)$, where \mathbf{A}_F denotes the ring of adèles of F . One approach to this problem is provided by the finite descent obstruction, which is defined to be the set of adelic points which can be lifted to twists of torsors for (certain) finite étale group schemes over F on X . More recently, Kim proposed an iterative construction of another subset of $X(\mathbf{A}_F)$ which contains the set of rational points. In this paper, we compare the two constructions. Our main result shows that the two approaches are essentially equivalent.

Keywords: Diophantine Geometry, Finite descent obstruction, Adelic points, Rational points

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*Imperial College London, South Kensington Campus, London SW7 2AZ;
otto.overkamp13@imperial.ac.uk

Introduction

Let F be a finite extension of the field \mathbf{Q} of rational numbers. Let X be a geometrically integral smooth algebraic variety over F with a rational point $b \in X(F)$, which we will keep fixed from now on. Let \overline{F} denote an algebraic closure of F and put $\overline{X} := X \times_F \text{Spec } \overline{F}$. Let us first describe the main ideas of this paper informally. Denote by \mathbf{A}_F the ring of adèles of F . One central question of Diophantine geometry is to locate precisely the set $X(F)$ inside $X(\mathbf{A}_F)$. There are several *local-to-global principles* that attempt to rectify the situation, for example the (finite) descent obstruction, introduced by Harari and Skorobogatov [HSk]. This is a set $X(\mathbf{A}_F)^{\text{f-cov}}$ such that

$$X(F) \subseteq X(\mathbf{A}_F)^{\text{f-cov}} \subseteq X(\mathbf{A}_F).$$

It is defined using finite étale covers of X ; a precise definition will be recalled later. On the other hand, we can construct similar sets lying between $X(F)$ and $X(\mathbf{A}_F)$ using ideas from homotopy theory and the geometric étale fundamental group $\pi_1(\overline{X}, b)$ of X . This group can be defined as the automorphism group of the functor Fib_b , which associates to a Galois covering of \overline{X} its fibre over b . In addition, if $x_v \in X(F_v)$ for any place v of F , we have *path torsors* $\pi_1(\overline{X}; b, x_v)$, elements of which are isomorphisms from Fib_b to Fib_{x_v} . This is clearly a right torsor for $\pi_1(\overline{X}, b)$. Furthermore, it carries a natural action of Γ_v which is compatible (in a suitable sense) with the action of Γ_v on $\pi_1(\overline{X}, b)$ that is inherited from the action of Γ_F on $\pi_1(\overline{X}, b)$. The path torsors $\pi_1(\overline{X}; b, x_v)$ thus define elements of the non-Abelian continuous cohomology set $H^1(\Gamma_v, \pi_1(\overline{X}, b))$. This way, we can construct a map

$$j: X(\mathbf{A}_F) \rightarrow \prod_v H^1(\Gamma_v, \pi_1(\overline{X}, b)),$$

which is usually called the *period map*. On the other hand, we also have a natural map

$$H^1(\Gamma_F, \pi_1(\overline{X}, b)) \rightarrow \prod_v H^1(\Gamma_v, \pi_1(\overline{X}, b)).$$

One sees easily that any adelic point P of X which comes from a rational point has the property that $j(P)$ lies in the image of this second map. By considering all adelic points of X which share this property, we can define another set which lies between $X(F)$ and $X(\mathbf{A}_F)$.

Recently, Kim [K] proposed an iterative construction of a subset $X(\mathbf{A}_F)_\infty$ of the set $X(\mathbf{A}_F)$ which contains the diagonal image of the set of rational points. His construction assumes that X has a rational point and some other mild conditions. In this paper, we interpret this construction in terms of the descent obstruction introduced by Harari and Skorobogatov [HS]. More precisely, we prove that the set $X(\mathbf{A}_F)_\infty$ is precisely the set of adelic points which survive finite descent with respect to finite nilpotent étale group schemes of odd order (Theorem 5.9).

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1 Twisting and the descent obstruction

In this section, let us introduce the *finite descent obstruction*. We will give a precise definition of the set $X(\mathbf{A}_F)^{\text{f-cov}}$, which has already been alluded to in the introduction. We will also introduce the main technical tools required for this, keeping the notation from the introduction. The main references for this section are [Sk] and [HSk].

Let $\Gamma_F := \text{Gal}(\overline{F}/F)$. Let G be a finite étale, but not necessarily commutative, group scheme over F . A *torsor* $Y \rightarrow X$ for G is a morphism $Y \rightarrow X$ of schemes which is fppf, and has a section locally in the fppf-topology, together with a right action of the X -group scheme $G \times_F X$ on Y over X such that the natural morphism $Y \times_F G \rightarrow Y \times_X Y$ is an isomorphism. Given two such torsors $Y \rightarrow X$ and $Z \rightarrow X$, we can construct the *contracted product* of Y and Z as follows: First, we construct an inverse Z^{-1} of Z . This is a *left* torsor for some inner form G' of G (see [Sk], p. 20, Example 2), whose underlying scheme is the same as that of Z , but with the action of G defined as $(g, z) \mapsto zg^{-1}$. Now we define the contracted product to be $Y \times^G Z^{-1}$ to be the quotient

$$Y \times^G Z^{-1} := (Y \times_X Z)/G,$$

where G acts diagonally (see [Sk], Lemma 2.2.3). We will mainly be interested in the case where Z arises from a torsor Φ for G over $\text{Spec } F$. In this case, Φ is given by a cocycle $\phi: \Gamma_F \rightarrow G(\overline{F})$ (see [Sk], (2.1)), and we define

$$Y^\phi := Y \times^G Z^{-1}.$$

This turns out to be a right torsor for a finite étale F -group scheme G^ϕ , which has the same \overline{F} -points as G and the same group structure, but the action of $\text{Gal}(\overline{F}/F)$ defining the F -scheme structure is given by $(\sigma, g) \mapsto \phi(\sigma)^\sigma g \phi(\sigma)^{-1}$. The group scheme G^ϕ is sometimes called an *inner twist* of G . If G is Abelian, then $G^\phi = G$. If $\pi: Y \rightarrow X$ denotes the structure morphism of some torsor, then the structure morphism of Y^ϕ will be denoted by π^ϕ . If ϕ_1 and ϕ_2 define the same class in $H^1(\Gamma_F, G)$, then the torsors Y^{ϕ_1} and Y^{ϕ_2} are isomorphic, albeit non-canonically so ([Sk], Chapter 2, (2.1)). We will sometimes abuse notation and write Y^ϕ for $\phi \in H^1(\Gamma_F, G)$ if no ambiguity can arise this way.

Definition 1.1 (Finite descent obstruction [S], Definition 5.4; see also [HSk], Definition 4.2)

$$X(\mathbf{A}_F)^{\text{f-cov}} := \bigcap_{Y \rightarrow X} \bigcup_{\tau \in H^1(\Gamma_F, G)} \pi^\tau(Y^\tau(\mathbf{A}_F)),$$

where the intersection is taken over torsors all finite étale group schemes G over F on X .

Now observe that, if $\Phi \rightarrow \operatorname{Spec} F$ is a torsor for G given by the cohomology class ϕ , then Φ^ϕ is trivial as a G^ϕ -torsor. Therefore, if $x: \operatorname{Spec} F \rightarrow X$ is a rational point, and $Y \rightarrow X$ is a torsor for some finite étale group scheme G , if τ denotes the cohomology class of the fibre Y_x (which is a torsor over $\operatorname{Spec} F$ for G), then the fibre of Y^τ over x has a rational point. This implies that $X(F) \subseteq X(\mathbf{A}_F)^{\text{f-cov}}$. Hence, the set $X(\mathbf{A}_F)^{\text{f-cov}}$ might provide some indication as to the location of $X(F)$ inside $X(\mathbf{A}_F)$. In this paper, we will need a slightly modified version of the descent obstruction:

Definition 1.2 *We say that an adelic point $(x_v)_v$ survives a torsor $\pi: Y \rightarrow X$ for some finite étale F -group scheme G if*

$$(x_v)_v \in \bigcup_{\sigma \in H^1(\Gamma_F, G)} \pi^\sigma(Y^\sigma(\mathbf{A}_F)).$$

Moreover, we define the set $X(\mathbf{A}_F)^{\text{Nil}_n}$ to be the set of adelic points of X which survive all finite étale group schemes G of odd order with $G(\overline{F})$ of nilpotency class $\leq n$, and define

$$X(\mathbf{A}_F)^{\text{Nil}} := \bigcap_{n \in \mathbf{N}} X(\mathbf{A}_F)^{\text{Nil}_n}.$$

The restriction that the order of G be odd has been added in the definition above only because it will make the statement of some theorems later more convenient.

2 Fundamental groups and path torsors

Let us keep the notation from the previous sections. In particular, let F be a number field. Let X a geometrically integral smooth algebraic variety over F and b a rational point of X , as before. For any field K containing F , any $x \in X(K)$, gives rise to a geometric point of \overline{X} , denoted also by x . Consider the *fibre functor at x*

$$F_x: \{\text{Finite étale covers of } \overline{X}\} \rightarrow \text{Sets} \\ (f: Y \rightarrow \overline{X}) \mapsto f^{-1}(x)(\overline{K}).$$

Then the *geometric étale fundamental group of \overline{X} with base point b* , denoted by $\pi_1(\overline{X}, b)$, is defined to be

$$\pi_1(\overline{X}, b) := \operatorname{Aut}(F_b)$$

(see [Sz], pp.172-173). Also, for any field K containing F , and $x \in X(K)$, define the *path torsor from b to x* , denoted by $\pi_1(\overline{X}; b, x)$ as the set of isomorphisms $F_b \rightarrow F_x$:

$$\pi_1(\overline{X}; b, x) := \operatorname{Isom}(F_b, F_x).$$

This set is non-empty ([Sz], Proposition 5.5.1), and it naturally carries the structure of a right $\pi_1(\overline{X}, b)$ -torsor. We can define the group $\pi_1(X, b)$ and the set $\pi_1(X; b, x)$ analogously. The étale fundamental group is functorial on pointed schemes ([Sz], p.178). We have an exact sequence

$$0 \rightarrow \pi_1(\overline{X}, b) \rightarrow \pi_1(X, b) \rightarrow \operatorname{Gal}(\overline{F}/F) \rightarrow 0,$$

(see [Sz], Proposition 5.6.1). By functoriality, the section $b: \operatorname{Spec} F \rightarrow X$ gives rise to a section $b_*: \operatorname{Gal}(\overline{F}/F) \rightarrow \pi_1(X, b)$, so we obtain an action of $\operatorname{Gal}(\overline{F}/F)$ on $\pi_1(\overline{X}, b)$ by conjugation. Now suppose that v is a place of F and choose a decomposition group $\operatorname{Gal}(\overline{F}_v/F_v) \subseteq \operatorname{Gal}(\overline{F}/F)$. Furthermore, let $x_v \in X(F_v)$. Then we obtain a map

$$x_{v*}: \operatorname{Gal}(\overline{F}_v/F_v) \rightarrow \pi_1(X \times_F \operatorname{Spec} F_v, x_v),$$

and similarly for $b \in X(F) \subseteq X(F_v)$. In particular, we obtain an action of $\operatorname{Gal}(\overline{F}_v/F_v)$ on $\pi_1(X \times_F \operatorname{Spec} F_v; b, x_v)$ using the map

$$\pi_1(X \times_F \operatorname{Spec} F_v, x_v) \times \pi_1(X \times_F \operatorname{Spec} F_v; b, x_v) \times \pi_1(X \times_F \operatorname{Spec} F_v, b) \rightarrow \pi_1(X \times_F \operatorname{Spec} F_v; b, x_v),$$

which descends to an action on $\pi_1(X \times_F \overline{F}_v; b, x_v) \cong \pi_1(\overline{X}; b, x_v)$. The existence of the canonical bijection of the path torsors follows because base extension gives rise to an equivalence of categories between the category of finite étale covers of \overline{X} and that of $X \times_F \operatorname{Spec} \overline{F}_v$. A direct calculation (see the Appendix, Section 7.1) shows that $\pi_1(\overline{X}; b, x_v)$ defines an element of the continuous cohomology set

$$H^1(\operatorname{Gal}(\overline{F}/F), \pi_1(\overline{X}, b)).$$

For more details about torsors and non-abelian cohomology, see [N], Proposition 1.2.3.

3 Kim's construction

Recently, Kim [K] defined a filtration

$$X(\mathbf{A}_F) = X(\mathbf{A}_F)_1 \supseteq X(\mathbf{A}_F)_1^2 \supseteq X(\mathbf{A}_F)_2 \supseteq \dots \supseteq X(\mathbf{A}_F)_n \supseteq X(\mathbf{A}_F)_n^{n+1} \supseteq \dots$$

If one sets

$$X(\mathbf{A}_F)_\infty := \bigcap_{n=1}^{\infty} X(\mathbf{A}_F)_n,$$

one finds that

$$X(F) \subseteq X(\mathbf{A}_F)_\infty.$$

If $X = \mathbf{G}_m$, the filtration is only non-trivial at the first level because the geometric fundamental group is Abelian, and can be constructed by means of global class field theory.

In this subsection, let us review this construction briefly. For more details, the reader should consult [K].

More precisely, the sets $X(\mathbf{A}_F)_{n-1}^n$ and $X(\mathbf{A}_F)_n$ are constructed as follows: Let $\pi_1(\overline{X}, b)$ be the étale fundamental group of \overline{X} with base point b and let $\Delta^{[1]}$ be its maximal prime-to-2 quotient. (One has to work with profinite groups which do not have any quotients of even order to deal with certain technical issues at the infinite places of F ; see [K], Proposition 2.1 and the comment thereafter.) Define inductively

$$\Delta^{[n+1]} := \overline{[\Delta^{[1]}, \Delta^{[n]}]}$$

and

$$\Delta_n := \Delta^{[1]} / \Delta^{[n+1]}.$$

Also define $T_n := \Delta^{[n]} / \Delta^{[n+1]}$, which leads to a central extension

$$0 \rightarrow T_n \rightarrow \Delta_n \rightarrow \Delta_{n-1} \rightarrow 0$$

for $n \geq 2$. The groups $\Delta^{[n]}$, Δ_n and T_n are all profinite, and T_n is Abelian for all $n \geq 1$.

If M denotes a finite set of odd primes, we denote by Δ_n^M the maximal pro- M quotient of Δ_n and similarly for T_n .

The groups Δ_n and Δ_n^M have the following property: Suppose G is a finite discrete group of nilpotency class $\leq n$. Then any continuous homomorphism $\Delta^{[1]} \rightarrow G$ factors through Δ_n . If, moreover, G has order divisible only by primes in M , then the morphism $\Delta^{[1]} \rightarrow G$ factors through Δ_n^M .

Let Γ_v be a decomposition group at v , which we identify with the absolute Galois group of F_v . From now on, we shall always assume that the conditions [Coh 1] (that T_n^M is torsion-free for all M) and [Coh 2] (that $H^0(\Gamma_v, T_n^M) = 0$ for all non-archimedean places v of F) are satisfied. These conditions are needed for technical reasons in Kim's construction to prove injectivity of some localization map and for related purposes (see [K], p. 5f), which is the only reason why we include them here. As observed in [K], they are always satisfied if X is a smooth curve.

Suppose S is a finite set of places of F . For fixed M , we call S *admissible* if the action of Γ_F on Δ_n^M factors through $\Gamma_F^S := \text{Gal}(F_S/F)$, where F_S is the largest extension of F unramified outside S . This condition is also required for technical reasons in this construction, in order to apply Poitou-Tate duality. One can show that such an S always exists, but the precise construction of such sets S for given M is rather involved; see [K], p.4. In any case, we will not use any facts about these sets other than their existence.

Recall that, for each n and M , we have a *period map*

$$j_n^M : X(\mathbf{A}_F) \rightarrow \prod'_v H^1(\Gamma_v, \Delta_n^M)$$

given by

$$(x_v)_v \mapsto ([\pi_1(\overline{X}; b, x_v) \times^{\pi_1(\overline{X}, b)} \Delta_n^M])_v$$

(see [K], Chapter 3). Here, the restricted product is taken with respect to the subsets $H^1(\Gamma_v/I_v, (\Delta_n^M)^{I_v})$ of $H^1(\Gamma_v, \Delta_n^M)$, where I_v is the inertia group at v . This makes sense because if for some place v of F , we have $x_v \in X(\mathfrak{o}_{F_v})$, then the element $\pi_1(\overline{X}; b, x_v)$ actually lies in $H^1(\Gamma_v/I_v, (\Delta_n^M)^{I_v})$ ([K], p.10). Also recall the localization map

$$H^1(\Gamma_F^S, \Delta_n^M) \rightarrow \prod'_v H^1(\Gamma_v, \Delta_n^M),$$

which is injective by [K], (2.17).

We define $X(\mathbf{A}_F)_1 := X(\mathbf{A}_F)$. First, one constructs a map

$$prec_1 : \varprojlim_M \prod'_v H^1(\Gamma_v, \Delta_1^M) \rightarrow H^1(\Gamma_F, D(\Delta_1))^\vee$$

and defines

$$X(\mathbf{A}_F)_1^2 := (prec_1 \circ j_1)^{-1}(0),$$

where $j_n := \varprojlim j_n^M$. Here, $D(-)$ refers to the continuous Galois dual $\text{Hom}_{\text{cont}}(-, \mu_\infty)$ and $-^\vee$ refers to the Pontriagin dual $\text{Hom}_{\text{cont}}(-, \mathbf{Q}/\mathbf{Z})$. The kernel of the map $prec_1$ is precisely $\varprojlim_M \varinjlim_S H^1(\Gamma_F^S, \Delta_1^M)$ by Poitou-Tate duality (See [K], Proposition 2.1 and [N], Theorem 8.6.7). Now, we have a connecting map

$$prec_1^2 := \delta_1^g : \varprojlim_M \varinjlim_S H^1(\Gamma_F^S, \Delta_1^M) \rightarrow \varprojlim_M \varinjlim_S H^2(\Gamma_F^S, T_2^M)$$

coming from the central extension above. The set $X(\mathbf{A}_F)_2$ is now defined to be the set of all elements of $X(\mathbf{A}_F)_1^2$ on which the composition $prec_1^2 \circ j_1 : X(\mathbf{A}_F)_1^2 \rightarrow \varprojlim_M \varinjlim_S H^2(\Gamma_F^S, T_2^M)$ vanishes. Now consider the projection map

$$p_1 : \prod_v' H^1(\Gamma_v, \Delta_2^M) \rightarrow \prod_v' H^1(\Gamma_v, \Delta_1^M).$$

The kernel of the connecting map

$$\varinjlim_S H^1(\Gamma_F^S, \Delta_1^M) \rightarrow \varinjlim_S H^2(\Gamma_F^S, T_2^M)$$

can be viewed as a subset of the target of p_1 , and we define $W(\Delta_2^M)$ to be the inverse image of this kernel under p_1 . Now one defines a map

$$prec_2 : \varprojlim_M W(\Delta_2^M) \rightarrow H^1(\Gamma_F, D(T_2))^\vee.$$

Since the image of $X(\mathbf{A}_F)_2$ under the period map j_2 is contained in $\varprojlim_M W(\Delta_2^M)$, we can now define

$$X(\mathbf{A}_F)_2^3 := (prec_2 \circ j_2)^{-1}(0).$$

From here, one proceeds by induction: Define

$$prec_n^{n+1} := \delta_n^g : \varprojlim_M \varinjlim_S H^1(\Gamma_F^S, \Delta_n^M) \rightarrow \varprojlim_M \varinjlim_S H^2(\Gamma_F^S, T_{n+1}^M),$$

and let $X(\mathbf{A}_F)_n$ be the set of elements of $X(\mathbf{A}_F)_{n-1}^n$ on which the composition

$$X(\mathbf{A}_F)_{n-1}^n \rightarrow \varprojlim_M \varinjlim_S H^1(\Gamma_F^S, \Delta_n^M) \rightarrow \varprojlim_M \varinjlim_S H^2(\Gamma_F^S, T_{n+1}^M)$$

vanishes. This makes sense by construction of $X(\mathbf{A}_F)_{n-1}^n$. Now one defines $W(\Delta_n^M)$ in a way entirely analogous to the construction of $W(\Delta_1^M)$ above, constructs a map

$$prec_{n+1} : \varprojlim_M W(\Delta_{n+1}^M) \rightarrow H^1(\Gamma_F, D(T_{n+1}))^\vee,$$

and defines

$$X(\mathbf{A}_F)_n^{n+1} := (prec_n \circ j_n)^{-1}(0).$$

The reader will have noticed several double limits of the form

$$\varprojlim_M \varinjlim_S H^1(\Gamma_F^S, \Delta_n^M).$$

They are to be interpreted in the following way: For fixed M , one considers the set of all admissible sets S of places of F , ordered by inclusion, and calculates the limit $\varinjlim_S H^1(\Gamma_F^S, \Delta_n^M)$. These sets naturally form a projective system, indexed by M (with the set of all M s also ordered by inclusion), and one can now make sense of the limit $\varprojlim_M (\varinjlim_S H^1(\Gamma_F^S, \Delta_n^M))$.

4 The Galois action on the geometric étale fundamental group

Consider the homotopy exact sequence

$$0 \rightarrow \pi_1(\overline{X}, b) \rightarrow \pi_1(X, b) \rightarrow \mathrm{Gal}(\overline{F}/F) \rightarrow 0.$$

Then the rational point $b: \mathrm{Spec} F \rightarrow X$ gives rise to a section $b_*: \mathrm{Gal}(\overline{F}/F) \rightarrow \pi_1(X, b)$, so $\mathrm{Gal}(\overline{F}/F)$ operates on $\pi_1(\overline{X}, b)$ by conjugation. This action plays a fundamental role in this paper, and we shall now try and describe it more explicitly. After recalling the description of $\pi_1(\overline{X}, b)$ as a profinite group, we will describe the Galois action on the finite quotients of $\pi_1(\overline{X}, b)$. Our goal is to find a description of the Galois action that is analogous to the description of the Galois action on the Tate module of an elliptic curve.

Let us now recall the profinite structure of $\pi_1(\overline{X}, b)$. Let $\{Y^\alpha\}$ be the set of all Galois covers $Y^\alpha \rightarrow \overline{X}$. Note that such a Galois cover Y^α is nothing but a connected right torsor for a finite étale group scheme G_α over \overline{F} . We define a partial ordering on the index set by putting $\beta \leq \alpha$ if there is an \overline{X} -morphism $\phi_{\alpha\beta}: Y^\alpha \rightarrow Y^\beta$. Furthermore, we assume that a system of such maps and geometric points $p_\alpha \in Y_b^\alpha(\overline{F})$, satisfying $\phi_{\alpha\beta}(p_\alpha) = p_\beta$, has been chosen. Note that for $\beta \leq \alpha$, there is at most one map satisfying this condition. Such a system together with a choice of geometric points can always be constructed, albeit in a non-canonical way (see [Sz], Chapter 5.4). For each α , we have a canonical isomorphism

$$\mathrm{Aut}_{\overline{X}}(Y^\alpha)^{\mathrm{op}} \cong G_\alpha(\overline{F})$$

(we have to consider the opposite group here because G_α acts from the right on Y^α , whereas, by convention, an automorphism group always acts from the left). For $\beta \leq \alpha$ the maps $p_{\alpha\beta}$ induce maps

$$\mathrm{Aut}_{\overline{X}}(Y^\alpha) \rightarrow \mathrm{Aut}_{\overline{X}}(Y^\beta),$$

which, in turn, give rise to an isomorphism

$$\pi_1(\overline{X}, b) \cong \varprojlim_\alpha \mathrm{Aut}_{\overline{X}}(Y^\alpha)^{\mathrm{op}} \cong \varprojlim_\alpha G_\alpha(\overline{F}).$$

(For more details about this construction, see [Sz], Chapter 5.4). This isomorphism explains why the system Y^α is referred to as a *pro-representing system*.

Now suppose that, for some α , Y^α arises from a geometrically connected torsor $Y \rightarrow X$ for some finite étale F -group scheme G . Then $G(\overline{F})$ automatically carries an action of $\text{Gal}(\overline{F}/F)$, and we obtain an induced homomorphism of topological groups

$$\pi: \pi_1(\overline{X}, b) \rightarrow G(\overline{F}).$$

In general, this homomorphism will not be equivariant, since the Galois action on the finite group $G(\overline{F})$ is not determined uniquely by the base change of the torsor $Y \rightarrow X$ to \overline{X} . As it turns out, however, a small adjustment will suffice to deal with this problem.

Lemma 4.1 *Let Y be a geometrically connected right torsor for G . Let $\tau: \Gamma_F \rightarrow G$ be the cocycle of the class of the fibre Y_b in $H^1(\Gamma_F, G)$ specified by*

$${}^\sigma y = y \cdot \tau(\sigma),$$

where $y \in Y_b(\overline{F})$ is the distinguished geometric point coming from the pro-representing system specified in Section 4. Then the map

$$\pi: \pi_1(\overline{X}, b) \rightarrow G^\tau$$

is Γ_F -equivariant.

Proof. The map π is specified by the requirement that

$$\gamma(y) = y \cdot \pi(\gamma)$$

for all $\gamma \in \pi_1(\overline{X}, b)$. Choosing a different distinguished geometric point $y' \in Y_b(\overline{F})$ gives rise to a new map

$$\hat{\pi}: \pi_1(\overline{X}, b) \rightarrow G.$$

Now let $\sigma \in \Gamma_F$ and suppose that $y' = {}^{\sigma^{-1}}y$. Then π and $\hat{\pi}$ are related by $\pi({}^\sigma \gamma) = {}^\sigma \hat{\pi}(\gamma)$. We compute

$$\begin{aligned} y \cdot \tau(\sigma) {}^\sigma \pi(\gamma) \tau(\sigma)^{-1} &= {}^\sigma y \cdot {}^\sigma \pi(\gamma) \tau(\sigma)^{-1} \\ &= {}^\sigma (y \cdot \pi(\gamma)) \cdot \tau(\sigma)^{-1} \\ &= {}^\sigma \gamma(y) \cdot \tau(\sigma)^{-1} \\ &= {}^\sigma \gamma({}^{\sigma^{-1}}y \cdot \tau(\sigma^{-1})^{-1}) \cdot \tau(\sigma)^{-1} \\ &= {}^\sigma \gamma({}^{\sigma^{-1}}y) \\ &= y \cdot \pi({}^\sigma \gamma), \end{aligned}$$

where the penultimate equality is due to the fact that γ is an automorphism of the fibre functor, and hence commutes with the right action of G . We also used the cocycle condition several times. The claim follows from this calculation. \square

We are now in a position to describe the Galois action on finite quotients of $\pi_1(\overline{X}, b)$, provided they arise as in Lemma 4.1. The remainder of this section is dedicated to showing that understanding these finite quotients suffices to understand the Galois action on $\pi_1(\overline{X}, b)$.

Definition 4.2 (Compare Stoll [S], p.19) *Let \mathcal{P} be a property of finite groups. A cofinal family of torsors with property \mathcal{P} is a family $\{Y^i \rightarrow X\}$ of geometrically connected torsors for finite étale group schemes G_i which have \mathcal{P} such that for all connected torsors $Z \rightarrow \overline{X}$ for a finite group that has \mathcal{P} , there is a map of torsors $\overline{Y}^i \rightarrow Z$ for some i .*

Recall that the finite étale F -group scheme G has property \mathcal{P} if the finite group $G(\overline{F})$ does. The properties \mathcal{P} that will be important for us will always be such that if a finite group G has \mathcal{P} and $H \subseteq G$ is a subgroup, then H has \mathcal{P} , and if G and H are finite groups both of which have \mathcal{P} , then $G \times H$ has \mathcal{P} .

Lemma 4.3 *Suppose that \mathcal{P} is such that*

- (i) *whenever K/F is a finite extension and G a finite étale group scheme over K which has \mathcal{P} , then $\text{Res}_{K/F} G$ has \mathcal{P} , and*
- (ii) *if G has \mathcal{P} then so does every subgroup scheme of G .*

Then there is a cofinal family of torsors with property \mathcal{P} . In particular, there exists a cofinal family of torsors under group schemes of odd order and of nilpotency class $\leq n$, for all n .

Proof. Let $Z \rightarrow \overline{X}$ be a connected right torsor for some finite group G which has \mathcal{P} . Then Z is, in fact, defined over some finite extension K of F , i.e. Z arises as the extension of scalars of some geometrically connected right torsor $Z_K \rightarrow X \times_F \text{Spec } K$. Following the proof of Stoll [S], Lemma 5.7, we obtain a geometrically connected right torsor

$$\text{Res}_{K/F} Z_K \rightarrow \text{Res}_{K/F}(X \times_F \text{Spec } K)$$

for the group scheme $\text{Res}_{K/F} G$. By assumption, this still has \mathcal{P} . Pulling this back to X via the canonical morphism $X \rightarrow \text{Res}_{K/F}(X \times_F \text{Spec } K)$, we obtain a torsor $Y \rightarrow X$ for $\text{Res}_{F/K} G$. From this, we obtain the diagram

$$\begin{array}{ccccc} Y \times_F \text{Spec } K & \longrightarrow & (\text{Res}_{K/F} Z_K) \times_F \text{Spec } K & \longrightarrow & Z_K \\ \downarrow & & \downarrow & & \downarrow \\ X \times_F \text{Spec } K & \longrightarrow & (\text{Res}_{K/F}(X \times_F \text{Spec } K)) \times_F \text{Spec } K & \longrightarrow & X \times_F \text{Spec } K, \end{array}$$

the two right horizontal arrows being the canonical maps, which come from the adjoint property of extension of scalars and Weil restriction. In particular, $Y \times_F \text{Spec } \overline{F}$ maps to Z . After possibly twisting Y , we may assume that Y lifts the rational point $b \in X(F)$. Hence, a connected component C of this twist will have a rational point, which will therefore be geometrically connected. This connected component is a torsor for its stabilizer, which still has property \mathcal{P} by assumption. Since \overline{C} visibly maps to Z , the claim follows. \square

We have already observed that the topological group $\pi_1(\overline{X}, b)$ is isomorphic to the inverse limit of the (opposite groups of) the automorphism groups of the Y^α . The previous lemma shows that we can also understand the action of $\text{Gal}(\overline{F}/F)$ on $\pi_1(\overline{X}, b)$ in profinite terms: Suppose $Y \rightarrow X$ and $Z \rightarrow X$ are geometrically connected right torsors for some finite étale group schemes G and H over F . They induce Galois

covers $\overline{Y} \rightarrow \overline{X}$ and $\overline{Z} \rightarrow \overline{X}$ with Galois groups $G(\overline{F})^{\text{op}}$ and $H(\overline{F})^{\text{op}}$, respectively. Suppose that $\overline{Z} \leq \overline{Y}$ with respect to the ordering on the set of all Galois covers of \overline{X} introduced before. If we now define $\tau := [Y_b] \in H^1(\Gamma_F, G)$ and $\mu := [Z_b] \in H^1(H, G)$, we obtain a Γ_F -equivariant map $G^\tau \rightarrow H^\mu$. The inverse system that arises this way is cofinal with the inverse system that arises from the set of all Galois covers of \overline{X} . Hence we obtain

Corollary 4.4 *There is a Γ_F -equivariant isomorphism*

$$\pi_1(\overline{X}, b) \cong \varprojlim_i G_i^{\tau_i},$$

where $\{Y^i \rightarrow X\}$ is a cofinal family of torsors for finite étale group schemes over F (G_i being the structure group of Y^i) and the maps between them are induced from the pro-representing system specified in section 4. Analogously, there is a Γ_F -equivariant isomorphism

$$\Delta_n^M \cong \varprojlim_i G_i^{\tau_i},$$

where $\{Y^i \rightarrow X\}$ is a cofinal family of torsors for finite étale group schemes over F of nilpotency class $\leq n$ and of order divisible only by primes in M .

Example. Let E be an elliptic curve over F . From the self-duality of elliptic curves it follows easily that the system of Galois coverings of $\overline{E} := E \times_F \text{Spec } \overline{F}$ defined by

$$[n]: \overline{E} \rightarrow \overline{E}$$

for $n \in \mathbf{N}$ is cofinal for the system of all Galois coverings of \overline{E} . Now, the map $[n]: E \rightarrow E$ over F defines a (right) torsor for the finite étale group scheme $E[n]$ over F . Therefore, the system $[n]: E \rightarrow E$, where $n \in \mathbf{N}$, is a cofinal family of torsors in the sense of Definition 1.3. Since everything is Abelian in this example, no twisting is necessary, and we obtain the familiar isomorphism

$$\pi_1(\overline{E}, 0) \cong \prod_{p \text{ prime}} T_p(E) = \varprojlim E[n],$$

where the set \mathbf{N} of natural numbers is ordered by divisibility.

5 Reciprocity and the descent obstruction

Let $n \in \mathbf{N}$ and suppose that M is a finite set of odd primes. Then let Δ_n be the quotient of $\Delta^{[1]}$ (which is, by definition, the maximal prime-to-2-quotient of $\pi_1(\overline{X}, b)$) modulo the n -th object of the (closed) lower central series of $\pi_1(\overline{X}, b)$ and let Δ_n^M be the maximal pro- M quotient of Δ_n . Recall the map

$$j_n^M: X(\mathbf{A}_F) \rightarrow \prod' H^1(\Gamma_v, \Delta_n^M)$$

given by

$$(x_v)_v \mapsto ([\pi_1(\overline{X}; b, x_v)^{\pi_1(\overline{X}, b)} \times \Delta_n^M])_v$$

(see [K], Chapter 3). Here, we are using continuous non-Abelian cohomology. The next proposition shows how the non-Abelian reciprocity law fits into our framework:

Proposition 5.1 *The diagram*

$$\begin{array}{ccc} X(\mathbf{A}_F)_n^{n+1} & \longrightarrow & X(\mathbf{A}_F) \\ \downarrow & & \downarrow j_n \\ \varprojlim_M \varinjlim_S H^1(\Gamma_F^S, \Delta_n^M) & \longrightarrow & \varprojlim_M \prod'_v H^1(\Gamma_v, \Delta_n^M) \end{array}$$

is Cartesian.

Proof. We will prove this by induction on n . If $n = 1$, we have $X(\mathbf{A}_F)_n = X(\mathbf{A}_F)$ by definition. The set $X(\mathbf{A}_F)_1^2$ is defined to be the set of elements of $X(\mathbf{A}_F) = X(\mathbf{A}_F)_1$ on which the composition

$$X(\mathbf{A}_F) \xrightarrow{j_1} \varprojlim_M \prod'_v H^1(\Gamma_v, \Delta_1^M) \xrightarrow{prec_1} H^1(\Gamma_F, D(\Delta_1))^\vee$$

vanishes. But the kernel of $prec_1$ is exactly the image of $\varprojlim_M \varinjlim_S H^1(\Gamma_F^S, \Delta_1^M)$ by Poitou-Tate duality (see [K], p.8), so the claim follows in this case. Now assume that we know the statement for $n - 1$. Consider the commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & X(\mathbf{A}_F) \\ \downarrow & & \downarrow \\ \varprojlim_M \varinjlim_S H^1(\Gamma_F^S, \Delta_n^M) & \longrightarrow & \varprojlim_M \prod'_v H^1(\Gamma_v, \Delta_n^M) , \\ \downarrow & & \downarrow \\ \varprojlim_M \varinjlim_S H^1(\Gamma_F^S, \Delta_{n-1}^M) & \longrightarrow & \varprojlim_M \prod'_v H^1(\Gamma_v, \Delta_{n-1}^M) \end{array}$$

where P is the pullback of $X(\mathbf{A}_F)$ with respect to the middle horizontal arrow. By the induction hypothesis, we know that the pullback of $X(\mathbf{A}_F)$ along the bottom horizontal arrow is equal to $X(\mathbf{A}_F)_{n-1}^n$. This clearly contains P . Consider the composition

$$P \rightarrow \varprojlim_M \varinjlim_S H^1(\Gamma_F^S, \Delta_{n-1}^M) \xrightarrow{prec_{n-1}^n} \varprojlim_M \varinjlim_S H^2(\Gamma_F^S, T_n^M).$$

This composition vanishes because the first map factors through $\varprojlim_M \varinjlim_S H^1(\Gamma_F^S, \Delta_n^M)$. Hence we find $P \subseteq X(\mathbf{A}_F)_n$. However, by [K], Proposition 2.4, we have $\ker prec_n = \varprojlim_M \varinjlim_S H^1(\Gamma_F^S, \Delta_n^M)$. By definition, $X(\mathbf{A}_F)_n^{n+1}$ is the set of elements of $X(\mathbf{A}_F)_n$ on which $prec_n$ vanishes, so we find $P = X(\mathbf{A}_F)_n^{n+1}$. This implies the result. \square

Now let v be a place of F and let $x_v \in X(F_v)$. Then the local Galois group $\text{Gal}(\overline{F}_v/F_v) =: \Gamma_v \subseteq \Gamma_F$ acts on the right path torsor $\pi_1(\overline{X}, b, x_v)$ in a fashion compatible with the action on $\pi_1(\overline{X}, b)$.

Proposition 5.2 *Assume that $Y \rightarrow X$ is a geometrically connected right torsor for some finite étale F -group scheme G . Let $\tau: \Gamma_v \rightarrow G(\overline{F})$ be the cocycle given by ${}^\sigma y = y \cdot \tau(\sigma)$, where y is the distinguished geometric point of Y_b coming from the pro-representing system we specified in Section 4. Its class in $H^1(\Gamma_v, G)$ is $[Y_b]$. Then the image of $[\pi_1(\overline{X}; b, x_v)]$ in $H^1(\Gamma_v, G^\tau)$ under the map*

$$H^1(\Gamma_v, \pi_1(\overline{X}, b)) \rightarrow H^1(\Gamma_v, G^\tau)$$

is $[Y_{x_v} \times^G Y_b^{-1}]$.

Proof. Let $y \in Y_b(\overline{F})$ be the distinguished geometric point. Define a map of sets

$$\pi_1(\overline{X}; b, x_v) \times^{\pi_1(\overline{X}, b)} G^\tau \rightarrow Y_{x_v} \times^G Y_b^{-1}$$

by

$$(\gamma, g) \mapsto (\gamma(y \cdot g), y).$$

One checks directly that this is well-defined and commutes with the action of G^τ , using the functoriality of F_b . Galois equivariance also follows directly. Since Γ_v -equivariant maps of torsors are isomorphisms, the claim follows. \square

Proposition 5.3 *Let $(x_v)_v \in X(\mathbf{A}_F)^{n+1}$. Let $\pi: Y \rightarrow X$ be a finite geometrically connected étale covering which is a torsor for some finite étale group G , which we assume to be of nilpotency class $\leq n$ and of order divisible only by primes in M , for some finite set M of odd prime numbers. Then there is an element $\sigma \in H^1(\Gamma_F, G)$ such that*

$$(x_v)_v \in \pi^\sigma(Y^\sigma(\mathbf{A}_F)).$$

Proof. Because $(x_v)_v \in X(\mathbf{A}_F)^{n+1}$, the image of $(x_v)_v$ in $\prod'_v H^1(\Gamma_v, \Delta_n^M)$ actually lies in the image of

$$\varinjlim_S H^1(\Gamma_F^S, \Delta_n^M) \rightarrow \prod'_v H^1(\Gamma_v, \Delta_n^M)$$

by Proposition 5.1. The maps $H^1(\Gamma_F^S, \Delta_n^M) \rightarrow H^1(\Gamma_F, \Delta_n^M)$ are compatible, so they give rise to a map

$$\varinjlim_S H^1(\Gamma_F^S, \Delta_n^M) \rightarrow H^1(\Gamma_F, \Delta_n^M).$$

Furthermore, since $b \in X(F)$, there is $\tau \in H^1(\Gamma_F, G)$ such that $b \in \pi^\tau(Y^\tau(\mathbf{A}_F))$. Now, the twisted inner form G^τ of G can be seen as a group isomorphic to G with a Γ_F -action on it. The finite étale cover $Y^\tau \rightarrow X$ gives rise to a Galois equivariant map $\Delta_n^M \rightarrow G^\tau$, and hence maps $H^1(\Gamma_F, \Delta_n^M) \rightarrow H^1(\Gamma_F, G^\tau)$ and $H^1(\Gamma_v, \Delta_n^M) \rightarrow H^1(\Gamma_v, G^\tau)$. By

Proposition 5.2, we already know that the image of $[\pi_1(\overline{X}; b, x_v) \times^{\pi_1(\overline{X}, b)} \Delta_n^M]$ under this last map is $[Y_{x_v}^\tau]$. Because the diagram

$$\begin{array}{ccc} H^1(\Gamma_F, \Delta_n^M) & \longrightarrow & H^1(\Gamma_v, \Delta_n^M) \\ \downarrow & & \downarrow \\ H^1(\Gamma_F, G^\tau) & \longrightarrow & H^1(\Gamma_v, G^\tau) \end{array}$$

is commutative for all places v of F , we can define σ to be the image of $(x_v)_v$ in $H^1(\Gamma_F, G^\tau)$. Indeed, the image of σ in $H^1(\Gamma_v, G^\tau)$ is $[Y_{x_v}^\tau]$, so $(x_v)_v$ lifts to $(Y^\tau)^\sigma$. Now we can use the canonical bijection between $H^1(\Gamma_F, G^\tau)$ and $H^1(\Gamma_F, G)$ to deduce the result. \square

For the next lemma, we need the following

Claim. Let $i = 1, 2$ and let $Y_i \rightarrow X$ be torsors for finite étale group schemes G_i . Suppose that Y_1 is geometrically connected, and assume that there is a map of torsors $\bar{Y}_1 \rightarrow \bar{Y}_2$ over \bar{X} . Then there is a cocycle $\sigma: \Gamma_F \rightarrow G_2(\bar{F})$ such that the map $\bar{Y}_1 \rightarrow \bar{Y}_2$ descends to a map $Y_1 \rightarrow Y_2^\sigma$ over X .

Proof. See Stoll [S], Lemma 5.6. \square

Lemma 5.4 *Let $Y^i \rightarrow X$ be a cofinal family of torsors for finite étale group schemes of odd order and nilpotency class $\leq n$, which exists by Lemma 4.3. Let G_i be the structure group scheme of $Y^i \rightarrow X$. Then an adelic point of X is contained in $X(\mathbf{A}_F)^{\text{Nil}_n}$ if and only if it survives all Y^i .*

Proof. We follow the proof of Stoll [S], Lemma 5.7. One direction is obvious, so let us consider the other one. Suppose $Y \rightarrow X$ is any right torsor for some finite étale F -group scheme G satisfying the hypotheses of the Lemma. Then $\bar{Y} \rightarrow \bar{X}$ is a (not necessarily connected) right torsor for the finite group $G(\bar{F})$. Let Y_0 be any connected component of \bar{Y} , and let G_0 be its stabilizer. Then Y_0 is a connected torsor for G_0 , and G_0 still has odd order and nilpotency class $\leq n$. Hence, by assumption, there is a member Y^i of our cofinal family such that we have a map $\bar{Y}^i \rightarrow Y_0 \rightarrow \bar{Y}$ over \bar{X} . By the claim above the lemma, there is a twist Y^σ of Y such that we obtain a morphism $Y^i \rightarrow Y^\sigma$. Now consider any adelic point P of X . By assumption, P will survive Y^i . Hence it will also survive Y . \square

Corollary 5.5 *We have $X(\mathbf{A}_F)_n^{n+1} \subseteq X(\mathbf{A}_F)^{\text{Nil}_n}$. In particular,*

$$X(\mathbf{A}_F)_\infty \subseteq X(\mathbf{A}_F)^{\text{Nil}}.$$

Now suppose that we have an adelic point $(x_v)_v \in X(\mathbf{A}_F)$ with the property that for all torsors $\pi: Y \rightarrow X$ for some finite étale group scheme G (of nilpotency class $\leq n$ and of order divisible only by primes in M) there exists an element $\sigma \in H^1(\Gamma_F, G)$ such that $(x_v)_v \in \pi^\sigma(Y^\sigma(\mathbf{A}_F))$. In order to show that $(x_v)_v \in X(\mathbf{A}_F)_n^{n+1}$, it suffices to show that the family of local classes $[\pi_1(\bar{X}; b, x_v) \times^{\pi_1(\bar{X}, b)} \Delta_n^M] \in \prod_v H^1(\Gamma_v, \Delta_n^M)$ comes from a global class, i.e. from an element of $H^1(\Gamma_F, \Delta_n^M)$:

Lemma 5.6 *Let $(x_v)_v \in X(\mathbf{A}_F)$. Let $Y \rightarrow X$ be a geometrically connected torsor for a finite étale group scheme G . Let $\tau := [Y_b] \in H^1(\Gamma_F, G)$ and suppose that there is a class $\mu \in H^1(\Gamma_F, G)$ such that $(x_v)_v$ lifts to Y^μ . Then the image of $([\pi_1(\bar{X}; b, x_v)])_v$ in $\prod_v H^1(\Gamma_v, G^\tau)$ under the map*

$$\prod_v H^1(\Gamma_v, \pi_1(\bar{X}, b)) \rightarrow \prod_v H^1(\Gamma_v, G^\tau)$$

lies in the image of $H^1(\Gamma_F, G^\tau) \rightarrow \prod_v H^1(\Gamma_v, G^\tau)$.

Proof. Let v be a place of F . By Proposition 5.2, we already know that the image of $[\pi_1(\overline{X}; b, x_v)]$ in $H^1(\Gamma_v, G^\tau)$ is $[Y_{x_v}^G \times Y_b^{-1}]$. But this is the same as $[Y_{x_v}^{\mu G^\mu} \times (Y_b^\mu)^{-1}]$. By assumption, $Y_{x_v}^\mu$ has F_v -points, so the class

$$[G^\mu \times (Y_b^\mu)^{-1}] \in H^1(\Gamma_F, G^\tau)$$

will do. Note that μ does not depend on the place v . This implies the claim. \square

Proposition 5.7 *Let $n \in \mathbf{N}$. Suppose that $(x_v)_v \in X(\mathbf{A}_F)_n$ survives all torsors $Y \rightarrow X$ for finite étale group schemes G of nilpotency class $\leq n$ and of order divisible by primes in M . Then $j_n^M((x_v)_v)$ lies in the image of $H^1(\Gamma_F, \Delta_n^M) \rightarrow \prod_v H^1(\Gamma_v, \Delta_n^M)$.*

Proof. Let (Y^i, G_i) be a cofinal system of such torsors, which exists by Lemma 4.3. Define $\tau_i := [Y_b^i] \in H^1(\Gamma_F, G_i)$. Then we obtain an isomorphism $\Delta_n^M \cong \varprojlim G_i^{\tau_i}$ and hence an isomorphism

$$H^1(\Gamma_v, \Delta_n^M) \rightarrow \varprojlim H^1(\Gamma_v, G_i^{\tau_i})$$

for each place v of F (see Lemma 7.2). Then the image of $[\pi_1(\overline{X}; b, x_v)]$ in $H^1(\Gamma_v, \Delta_n^M)$ corresponds to the projective system $[Y_{x_v}^{G_i} \times (Y_b^i)^{-1}]$. For each i , the set S_i of preimages of $[(Y_{x_v}^{G_i} \times (Y_b^i)^{-1})]_v \in \prod_v H^1(\Gamma_v, G_i^{\tau_i})$ in $H^1(\Gamma_F, G_i^{\tau_i})$ is non-empty by Lemma 5.6, and it is finite by Lemma 7.1. Furthermore, the maps $H^1(\Gamma_F, G_j^{\tau_j}) \rightarrow H^1(\Gamma_F, G_k^{\tau_k})$ for $\overline{Y}^k \leq \overline{Y}^j$ descend to maps $S_j \rightarrow S_k$. It follows that the limit $\varprojlim S_i$ is non-empty. Any element of this limit can be viewed as an element of $H^1(\Gamma_F, \Delta_n^M)$. Hence the result follows. \square

Lemma 5.8 *Let S be an admissible finite set of places of F (i.e. the action of Γ_F on Δ_n^M factors through Γ_F^S). Let $\alpha \in H^1(\Gamma_F, \Delta_n^M)$ be such that its image in $H^1(\Gamma_v, \Delta_n^M)$ comes from $H^1(\Gamma_v/I_v, (\Delta_n^M)^{I_v})$ for all places $v \notin S$. Then α lies in the image of $H^1(\Gamma_F^S, \Delta_n^M) \rightarrow H^1(\Gamma_F, \Delta_n^M)$.*

Proof. Pick a cocycle representing the class $\alpha \in H^1(\Gamma_F, \Delta_n^M)$, and denote it by $\alpha: \Gamma_F \rightarrow \Delta_n^M$. By our assumption on α , we have $\alpha(I_v) = 0$ for all $v \notin S$. But since the groups I_v topologically generate the group $\ker(\Gamma_F \rightarrow \Gamma_F^S)$ (this follows, for example, from [Na], Chapter 6.1, Corollary 2, and a limit argument) and the cocycles in question are continuous, this implies that $\alpha(\ker(\Gamma_F \rightarrow \Gamma_F^S)) = 0$. Using that $\ker(\Gamma_F \rightarrow \Gamma_F^S)$ acts trivially on Δ_n^M since S is admissible, we find that α defines a class in $H^1(\Gamma_F^S, \Delta_n^M)$. This concludes the proof. \square

Theorem 5.9 *Let X be a geometrically integral smooth projective variety over F with a rational point b . Assume that X satisfies the conditions [Coh 1] and [Coh 2] from [K], p.3. Then we have $X(\mathbf{A}_F)^{\text{Nil}_n} = X(\mathbf{A}_F)_n^{n+1}$ for all $n \in \mathbf{N}$. In particular,*

$$X(\mathbf{A}_F)^{\text{Nil}} = X(\mathbf{A}_F)_\infty.$$

Proof. By Corollary 5.5, all we have to show is the inclusion " \subseteq ". Suppose $(x_v)_v \in X(\mathbf{A}_F)^{\text{Nil}_n}$. By Proposition 5.7 we already know that $j_n^M((x_v)_v)$ lies in the image of

$$H^1(\Gamma_F, \Delta_n^M) \rightarrow \prod_v H^1(\Gamma_v, \Delta_n^M)$$

for all finite sets M of odd prime numbers. However, we know that $j_n^M((x_v)_v)$ lies in the restricted product $\prod' H^1(\Gamma_v, \Delta_n^M)$. By [K], Chapter 2, we can find an admissible set S of places of F which contains all places v such that the image of $j_n^M((x_v)_v)$ in $H^1(\Gamma_v, \Delta_n^M)$ does *not* come from $H^1(\Gamma_v/I_v, (\Delta_n^M)^{I_v})$. From Lemma 5.8 it now follows that $j_n^M((x_v)_v)$ lies in the image of

$$\varinjlim H^1(\Gamma_F^S, \Delta_n^M) \rightarrow \prod' H^1(\Gamma_v, \Delta_n^M).$$

Now suppose $M \subseteq N$ are two finite sets of odd prime numbers. Consider the commutative diagram

$$\begin{array}{ccc} \varinjlim H^1(\Gamma_F^S, \Delta_n^N) & \longrightarrow & \prod' H^1(\Gamma_v, \Delta_n^N) \\ \downarrow & & \downarrow \\ \varinjlim H^1(\Gamma_F^S, \Delta_n^M) & \longrightarrow & \prod' H^1(\Gamma_v, \Delta_n^M). \end{array}$$

The horizontal arrows are injective because the localization maps

$$H^1(\Gamma_F^S, \Delta_n^M) \rightarrow \prod' H^1(\Gamma_v, \Delta_n^M)$$

are injective ([K], (2.17)). Hence, the preimage of $j_n^N((x_v)_v)$ in $\varinjlim_S H^1(\Gamma_F^S, \Delta_n^N)$ must be mapped to that of $j_n^M((x_v)_v)$ in $\varinjlim_S H^1(\Gamma_F^S, \Delta_n^M)$. This implies that the element

$$\varprojlim_M j_n^M((x_v)_v) \in \varprojlim_M \prod' H^1(\Gamma_v, \Delta_n^M)$$

lies in the image of the map

$$\varprojlim_M \varinjlim H^1(\Gamma_F^S, \Delta_n^M) \rightarrow \varprojlim_M \prod' H^1(\Gamma_v, \Delta_n^M).$$

□

Remark. The sets $X(\mathbf{A}_F)^{n+1}$ are only defined if X has a rational point and satisfies the conditions [Coh1] and [Coh2] which are required for the construction of the non-Abelian reciprocity law. However, the sets $X(\mathbf{A}_F)^{\text{Nil}_n}$ are defined without any restriction of this kind.

6 The descent obstruction revisited

This section is devoted to giving a new definition of the set $X(\mathbf{A}_F)^{\text{f-cov}}$, using some of the ideas from the previous sections. See also [HS] for related discussions. In this section, we assume that X is a geometrically integral smooth projective variety over F which has a rational point b . The conditions [Coh 1] and [Coh 2] from [K], p.3, which we always assumed in the previous sections, can now be omitted. Define Q to be the set defined by the Cartesian diagram

$$\begin{array}{ccc} Q & \longrightarrow & X(\mathbf{A}_F) \\ \downarrow & & \downarrow \\ H^1(\Gamma_F, \pi_1(\overline{X}, b)) & \longrightarrow & \prod_v H^1(\Gamma_v, \pi_1(\overline{X}, b)). \end{array}$$

Theorem 6.1 *The set $X(\mathbf{A}_F)^{\text{f-cov}}$ coincides with the image of $Q \rightarrow X(\mathbf{A}_F)$.*

Proof. Suppose $Y \rightarrow X$ is a geometrically connected (right) torsor for a finite étale group scheme G and let $\tau := [Y_b] \in H^1(\Gamma_F, G)$. By Lemma 4.1, we obtain a Γ_F -equivariant map

$$\pi_1(\overline{X}, b) \rightarrow G^\tau.$$

Let $(x_v)_v$ be contained in the image of $Q \rightarrow X(\mathbf{A}_F)$. Because the diagram

$$\begin{array}{ccc} H^1(\Gamma_F, \pi_1(\overline{X}, b)) & \longrightarrow & \prod_v H^1(\Gamma_v, \pi_1(\overline{X}, b)) \\ \downarrow & & \downarrow \\ H^1(\Gamma_F, G^\tau) & \longrightarrow & \prod_v H^1(\Gamma_v, G^\tau) \end{array}$$

is commutative, if we pick $\mu \in H^1(\Gamma_F, G^\tau)$ such that the images of $(x_v)_v$ and μ in $\prod_v H^1(\Gamma_v, G^\tau)$ coincide, we find that $(x_v)_v$ lifts to $(Y^\tau)^\mu$. On the other hand, suppose that $(x_v)_v \in X(\mathbf{A}_F)^{\text{f-cov}}$. Pick a cofinal system (Y^i, G_i) . Then, for any place v of F , the torsor $[\pi_1(\overline{X}; b, x_v)]$ is represented by the projective system

$$([Y_{x_v}^i \times^{G_i} (Y_b^i)^{-1}])_i \in \varprojlim_i H^1(\Gamma_v, G_i^{\tau_i}) = H^1(\Gamma_v, \pi_1(\overline{X}, b))$$

(see Lemma 7.2), where $\tau_i := [Y_b^i] \in H^1(\Gamma_v, G_i)$. For each i , the sequence $([Y_{x_v}^i \times^{G_i} (Y_b^i)^{-1}]) \in \prod_v H^1(\Gamma_v, G_i^{\tau_i})$ has a pre-image in $H^1(\Gamma_F, G_i^{\tau_i})$. Let S_j be the set of all such pre-images. Then S_j is finite for all j by Lemma 7.1 and the maps $H^1(\Gamma_F, G_i^{\tau_i}) \rightarrow H^1(\Gamma_F, G_j^{\tau_j})$ induce maps $S_i \rightarrow S_j$. It follows that the limit $\varprojlim_j S_j$ is non-empty by Lemma 5.6, so the result follows. \square

7 Appendix

In this appendix, we will prove a few lemmata about non-Abelian (continuous) cohomology which we have used in this note. The notation will be the same as before; in particular, F will denote a number field.

Lemma 7.1 (Minkowski-Hermite) *Let G be a finite discrete (not necessarily Abelian) group with a continuous action of Γ_F . Then the localization map*

$$H^1(\Gamma_F, G) \rightarrow \prod_v H^1(\Gamma_v, G)$$

has finite fibres.

Proof. See Stix [St], Lemma 146. □

Lemma 7.2 *Let $(G_i)_{i \in I}$ be a filtered projective system of finite discrete (not necessarily Abelian) groups with a continuous action of Γ_F . Then the natural map*

$$H^1(\Gamma_F, \varprojlim_i G_i) \rightarrow \varprojlim_i H^1(\Gamma_F, G_i)$$

is an isomorphism, where the left hand side denotes the continuous cohomology set.

Proof. For each $i \in I$, let Z^i denote the set of continuous G_i -valued cocycles of Γ_F , and let Z denote the set of continuous $\varprojlim_i G_i$ -valued cocycles of Γ_F . We obtain a bijection

$$Z \rightarrow \varprojlim_i Z^i.$$

Now, $H^1(\Gamma_F, G_i)$ is defined to be the quotient of Z^i modulo the action of G_i given by

$$(g \cdot \psi)(\sigma) := g \psi(\sigma) {}^\sigma g^{-1},$$

where $g \in G_i$ and $\psi \in Z^i$, and similarly for $\varprojlim_i G_i$. It is clear that if $\psi, \psi' \in Z$ lie in the same orbit modulo the action of $\varprojlim_i G_i$, then the images $\psi_i, \psi'_i \in Z^i$ lie in the same orbit modulo the action of G_i , so we obtain a well-defined map

$$H^1(\Gamma_F, \varprojlim_i G_i) \rightarrow \varprojlim_i H^1(\Gamma_F, G_i),$$

which is clearly surjective. To see that it is injective, suppose we have elements $(\psi_i)_{i \in I}$ and $(\psi'_i)_{i \in I}$ of $\varprojlim_i Z^i$ such that for each $i \in I$, the set S_i of all $g_i \in G_i$ with the property

$$\psi'_i(\sigma) = g_i \psi_i(\sigma) {}^\sigma g_i^{-1}$$

is non-empty. For $i \leq j \in I$, we obtain a map $S_j \rightarrow S_i$. Since the sets S_i are finite (because they are subsets of G_i), we find that the limit $\varprojlim_i S_i$ is non-empty, so the elements $\varprojlim_i \psi_i$ and $\varprojlim_i \psi'_i$ of Z lie in the same orbit. The result follows. □

7.1 The cohomology class of a path torsor

Let us now consider the path torsors $\pi_1(\overline{X}; b, x_v)$ which appeared multiple times in this text. A priori, for $b \in X(F)$ and $x_v \in X(F_v)$ (where v is a place of F), $\pi_1(\overline{X}; b, x_v)$ is defined to be the set of isomorphisms of fibre functors

$$I: F_b \rightarrow F_{x_v}.$$

Here, F_b denotes the functor from the category of finite étale covers of \overline{X} to the category of sets given by

$$(Y \rightarrow \overline{X}) \mapsto Y_b(\overline{F}),$$

and similarly for x_v . The path torsors carry an obvious right $\pi_1(\overline{X}, b)$ -action, as well as a left action of $\Gamma_v \subseteq \Gamma_F$. If $Y \rightarrow X$ is a finite étale cover of X (*not* of \overline{X}), the latter is given by

$$({}^\sigma I)(y) := {}^\sigma I(\sigma^{-1} y)$$

for $y \in Y_b(\overline{F})$, $I \in \pi_1(\overline{X}; b, x_v)$ and $\sigma \in \Gamma_v$. Here, we use the canonical injection

$$\pi_1(\overline{X}; b, x_v) \rightarrow \pi_1(X; b, x_v).$$

Note that $I(\sigma^{-1} y)$ is viewed as an element of $Y_{x_v}(\overline{F}_v)$, which is why we do not have an action of the full Galois group Γ_F on $\pi_1(\overline{X}; b, x_v)$, but only one of Γ_v . To see that this defines an element of $H^1(\Gamma_v, \pi_1(\overline{X}, b))$, pick any $I \in \pi_1(\overline{X}; b, x_v)$. Then, for any $\sigma \in \Gamma_v$, there is a unique element $\tau(\sigma) \in \pi_1(\overline{X}, b)$ such that

$${}^\sigma I = I\tau(\sigma).$$

A simple calculation shows that

$$\tau: \Gamma_v \rightarrow \pi_1(\overline{X}, b)$$

is a cocycle. Choosing a different I will give rise to a cohomologous cocycle. Hence, all that remains to be seen is that τ is continuous. Suppose $Y \rightarrow X$ is a geometrically connected right torsor for some finite étale F -group scheme G . Then we obtain a continuous homomorphism of topological groups

$$\pi: \pi_1(\overline{X}, b) \rightarrow G(\overline{F}),$$

the latter group being endowed with the discrete topology. Let $y \in Y_b(\overline{F}_v)$ and let $\mu, \nu: \Gamma_v \rightarrow G(\overline{F}_v)$ be the cocycles such that ${}^\sigma y = y \cdot \mu(\sigma)$ and ${}^\sigma I(y) = I(y) \cdot \nu(\sigma)$, respectively. Then we have

$$\begin{aligned} I(y \cdot \pi(\tau(\sigma))) &= ({}^\sigma I)(y) \\ &= {}^\sigma I(y \cdot \mu(\sigma^{-1})) \\ &= I(y) \cdot \nu(\sigma) \mu(\sigma)^{-1}. \end{aligned}$$

Therefore, $\pi(\tau(\sigma)) = \nu(\sigma) \mu(\sigma)^{-1}$, which is certainly continuous. Because the topology on $\pi_1(\overline{X}, b)$ is defined to be the profinite topology with respect to the various projections $\pi: \pi_1(\overline{X}, b) \rightarrow G(\overline{F})$, the continuity of τ follows.

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